Approximation by semigroup of spherical operators

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Abstract This paper concerns about the approximation by a class of positive exponential type multiplier operators on the unit sphere $\mathbb{S}^n$ of the $(n+1)$-dimensional Euclidean space for $n \geqslant 2$. We prove that such operators form a strongly continuous contraction semigroup of class $(\mathscr{C}_0)$ and show the equivalence between the approximation errors of these operators and the K-functionals. We also give the saturation order and the saturation class of these operators. As examples, the rth Boolean of the generalized spherical Abel-Poisson operator $\oplus^r V_t^\gamma$ and the rth Boolean of the generalized spherical Weierstrass operator $\oplus^r W_t^\kappa$ for integer $r \geqslant 1$ and reals $\gamma, \kappa \in (0,1]$ have errors $\| \oplus^r V_t^\gamma f - f\|_{\mathscr{X}} \asymp \omega^{r\gamma}(f, t^{1/\gamma})_{\mathscr{X}}$ and $\| \oplus^r W_t^\kappa f - f\|_{\mathscr{X}} \asymp \omega^{2r\kappa}(f, t^{1/(2\kappa)})_{\mathscr{X}}$ for all $f \in \mathscr{X}$ and $0 \leqslant t \leqslant 2\pi$, where $\mathscr{X}$ is the Banach space of all continuous functions or all $\mathscr{L}^p$ integrable functions, $1 \leqslant p < +\infty$, on $\mathbb{S}^n$ with norm $\|\cdot\|_{\mathscr{X}}$, and $\omega^s(f,t)_{\mathscr{X}}$ is the modulus of smoothness of degree $s > 0$ for $f \in \mathscr{X}$. Moreover, $\oplus^r V_t^\gamma$ and $\oplus^r W_t^\kappa$ have the same saturation class if $\gamma = 2\kappa$.

Keywords Sphere, semigroup, approximation, modulus of smoothness, multiplier

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1 Introduction

Let $\mathbb{S}^n$ be the unit sphere of $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. Denote by

$$\mathscr{L}^p(\mathbb{S}^n) := \mathscr{L}^p(\mathbb{S}^n, \sigma_n), \quad 1 \leqslant p < +\infty,$$

the complex-valued $\mathscr{L}^p$-function space with respect to the surface measure σ_n on $\mathbb{S}^n$, and let $\mathscr{C}(\mathbb{S}^n)$ be the space of all complex-valued continuous functions. Denote by $|\mathbb{S}^n| := \sigma_n(\mathbb{S}^n)$ the volume of $\mathbb{S}^n$. Any square integrable function f

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on $\mathbb{S}^n$ admits the Fourier-Laplace expansion

$$f=\sum_{k=0}^{+\infty} Y_k(f),$$

where, letting $\lambda := (n-1)/2$,

$$Y_k(f;\boldsymbol{x}) := \frac{\Gamma(\lambda)(k+\lambda)}{2\pi^{\lambda+1}} \int_{\mathbb{S}^n} C_k^{(\lambda)}(\boldsymbol{x}\cdot\boldsymbol{y}) f(\boldsymbol{y})\mathrm{d}\sigma_n(\boldsymbol{y}) \tag{1.1}$$

is the projection of f onto the space Π_k^n of all spherical polynomials of degree at most k, where $C_k^{(\lambda)}(s)$ $(-1\leqslant s\leqslant 1)$ is the Gegenbauer polynomial of degree k with λ. The Gegenbauer polynomials $C_k^{(\lambda)}(s)$ $(k=0,1,2,\dots)$ are generated by

$$\frac{1}{(1-2su+u^2)^{\lambda}} = \sum_{k=0}^{+\infty} C_k^{(\lambda)}(s)u^k, \quad 0\leqslant u<1. \tag{1.2}$$

Letting $u=\mathrm{e}^{-t}$ for $t>0$, by (1.2) and (1.1), we have

$$V_t^1(f;\boldsymbol{x}) := \sum_{k=0}^{+\infty} u^k Y_k(f;\boldsymbol{x}) = \frac{1}{|\mathbb{S}^n|}\int_{\mathbb{S}^n} \frac{(1-u^2)f(\boldsymbol{y})\mathrm{d}\sigma_n(\boldsymbol{y})}{(1-2u(\boldsymbol{x}\cdot\boldsymbol{y})+u^2)^{\lambda+1}}. \tag{1.3}$$

Formula (1.3) defines the Abel-Poisson sum of f of $\mathscr{L}^1(\mathbb{S}^n)$. The convergence of (1.2) is guaranteed by the convergence of (1.3).

Let B be a Banach space, and let T be an operator on B. Then the K*-functional* induced by T is given by

$$K_T(f,t)_B := \inf\{\|f-g\|_B + t\|T(g)\|_B \colon T(g)\in B\}. \tag{1.4}$$

Dai and Ditzian [8] proved the equivalence between the approximation error of the Abel-Poisson sum and the K-functional. In more details, let $\mathscr{X}$ be either the $\mathscr{L}^p(\mathbb{S}^n)$ or $\mathscr{C}(\mathbb{S}^n)$, and let $\mathscr{V}$ be the operator on $\mathscr{X}$ such that its eigenvalues are $-k$ for Π_k^n, $k=0,1,2,\dots$. Then,

$$\|V_t^1(f)-f\|_{\mathscr{X}} \asymp K_{\mathscr{V}}(f,t)_{\mathscr{X}}. \tag{1.5}$$

This equivalence shows the approximation capability of the Abel-Poisson sum for the $\mathscr{L}^p$ and the continuous spaces in terms of the K-functional.

Let $q=q(x)$ be a polynomial and $0<\gamma\leqslant 1$. In this paper, we generalize (1.5) to a class of *exponential type operators* $\{T_{q,t}^{\gamma}\colon 0\leqslant t<+\infty\}$ that have Fourier-Laplace expansion of

$$T_{q,t}^{\gamma}(f) \sim \sum_{k=0}^{+\infty} \mathrm{e}^{-(q(k))^{\gamma}t} Y_k(f), \quad t>0, \tag{1.6}$$

and we let

$$T^{\gamma}_{q,0}(f) := f. \tag{1.7}$$

We say that $T^{\gamma}_{q,t}$ is *regular* if the degree of $q(x)$ and the coefficient of the first term are both positive and $q(0) = 0$. An operator T on $\mathscr{X}$ is said to be *positive* if $T(f) \geqslant 0$ whenever $f \geqslant 0$. We note that when $q(x) = x$ and $\gamma = 1$, $T^{\gamma}_{q,t}$ reduces to the Abel-Poisson sum of (1.3). The regular and positive $T^{\gamma}_{q,t}$ form a special semigroup with respect to t, as follows.

For a Banach space B, let $\{T_t \colon t \geqslant 0\}$ be a semigroup of operators on B. We say that T_t strongly converges to M in B as $t \to t_0$ if the operator norm

$$\|T_t - M\|_{B\to B} \to 0, \quad t \to t_0$$

(the limit may be one-sided). We denote the limit by

$$M := s\text{-}\lim_{t\to t_0} T_t.$$

Let I be the identity operator. The semigroup T_t is said to be a (strongly continuous) *semigroup of class* $(\mathscr{C}_0)$ if we have the following two conditions:

$$T_0 = I, \quad T_{t_1+t_2} = T_{t_1}T_{t_2}, \quad t_1, t_2 \geqslant 0, \tag{1.8a}$$

$$s\text{-}\lim_{t\to 0+} T_t = I. \tag{1.8b}$$

We say that T_t is a contraction semigroup if for $t > 0$ and $f \in B$,

$$\|T_t f\|_B \leqslant \|f\|_B.$$

The *infinitesimal generator* $\mathscr{A}$ of the semigroup $\{T_t \colon 0 \leqslant t < +\infty\}$ is defined by (see [6, p. 11])

$$\mathscr{A} f := s\text{-}\lim_{t\to 0+} \frac{T_t f - f}{t},$$

whenever the limit exists; the domain of $\mathscr{A}$ is, in symbols $\mathscr{D}(\mathscr{A})$, being the set of elements $f \in \mathscr{X}$ for which the limit exists; for $r = 0, 1, 2, \dots$, the rth power $\mathscr{A}^r$ of the infinitesimal generator $\mathscr{A}$ is defined inductively by the relations

$$\mathscr{A}^0 = I, \quad \mathscr{A}^1 = \mathscr{A},$$

and

$$\mathscr{D}(\mathscr{A}^r) := \{f \colon f \in \mathscr{D}(\mathscr{A}^{r-1}),\ \mathscr{A}^{r-1}f \in \mathscr{D}(\mathscr{A})\}, \tag{1.9a}$$

$$\mathscr{A}^r f := \mathscr{A}(\mathscr{A}^{r-1}f) = s\text{-}\lim_{t\to 0+} \frac{T_t - I}{t} \mathscr{A}^{r-1} f, \quad f \in \mathscr{D}(\mathscr{A}^r). \tag{1.9b}$$

See, e.g., [6, pp. 7, 8].

In Theorem 1 (Section 4), we prove that if the operator $T^{\gamma}_{q,t}$ is regular and positive for each $t \in [0, +\infty)$, then $\{T^{\gamma}_{q,t} \colon 0 \leqslant t < +\infty\}$ form a strongly continuous contraction semigroup of class ($\mathscr{C}_0$). Let $\mathscr{A}_q^\gamma$ be the operator on $\mathscr{X}$ such that $\mathscr{A}_q^\gamma$ has eigenvalues $-(q(k))^\gamma$ for Π_k^n, $k=0,1,2,\ldots$. Then $\mathscr{A}_q^\gamma$ is the infinitesimal generator of the semigroup $T_{q,t}^\gamma$. For integer $r \geqslant 1$, the rth power $(\mathscr{A}_q^\gamma)^r$ of $\mathscr{A}_q^\gamma$ is the operator on $\mathscr{X}$ that has eigenvalues of $(-(q(k))^\gamma)^r$ for Π_k^n. We prove the following Bernstein type inequality for the semigroup $T_{q,t}^\gamma$, see Theorem 1.

Main Theorem—Bernstein type inequality *Let $T_{q,t}^\gamma$ defined above be regular and positive. Then*

$$\|\mathscr{A}_q^\gamma T_{q,t}^\gamma f\|_{\mathscr{X}} \leqslant \frac{c}{t}\,\|f\|_{\mathscr{X}}, \tag{1.10}$$

where the constant c depends only on n, q, and $\mathscr{X}$.

Ditzian and Ivanov [9] proved that the approximation error of the seimgroup of class ($\mathscr{C}_0$) is equivalent to the K-functional induced by the infinitesimal generator of the semigroup if the Bernstein type inequality holds for the semigroup. For positive integer r, the rth *Boolean* (*sum*) *of an operator* T on a Banach space B is defined by

$$\oplus^r T := I-(I-T)^r = -\sum_{i=1}^{r}(-1)^i\binom{r}{i}T^i, \tag{1.11}$$

and we let $T^0 := I$. Ditzian and Ivanov [9] showed that if there exists some constant c independent of t and f such that

$$t\|\mathscr{A}T_t f\|_B \leqslant c\|f\|_B,$$

then for $r\in\mathbb{Z}_+$,

$$\|\oplus^r T_t f - f\|_{\mathscr{X}} \asymp K_{\mathscr{A}^r}(f,t^r)_B, \quad \forall\ f\in B,\ \forall\ t\geqslant 0.$$

By this way, we may obtain from (1.10) that the approximation error of the rth Boolean of $T_{q,t}^\gamma$ is equivalent to the K-functional induced by $\mathscr{A}_{q^r}^\gamma$, see Theorem 2 in Section 4.

Main Theorem—Approximation error *Let $T_{q,t}^\gamma$ defined above be regular and positive, and let r be a positive integer. Then*

$$\|f-\oplus^r T_{q,t}^\gamma(f)\|_{\mathscr{X}} \asymp K_{(\mathscr{A}_q^\gamma)^r}(f,t^r)_{\mathscr{X}}, \quad f\in\mathscr{X}, \tag{1.12}$$

where the constants in the inequalities depend only on n, q, γ, and r.

Let $\phi(\rho)$ be a positive function with respect to ρ, $0<\rho<+\infty$, which converges monotonically to zero as $\rho\to+\infty$. For a sequence of operators $\{I_\rho\}_{\rho>0}$ on $\mathscr{X}$, assume that there exists $\mathscr{K}\subseteqq\mathscr{X}$ such that

(i) if $\|I_\rho(f)-f\|_{\mathscr{X}} = o(\phi(\rho))$, then $I_\rho f = f$ for all $\rho>0$;

(ii) $\|I_\rho(f)-f\|_{\mathscr{X}} = \mathscr{O}(\phi(\rho))$ if and only if $f\in\mathscr{K}$.

Then we say I_ρ are *saturated* on $\mathscr{X}$ with order $\mathscr{O}(\phi(\rho))$ and have $\mathscr{K}$ as their *saturation class*. We refer the reader to [3, p. 217] and [11].

For each $t > 0$, $\oplus^r T^\gamma_{q,t}$ is a positive operator, see the proof of Theorem 2 below. The Booleans $\oplus^r T^\gamma_{q,t}$, $t \geqslant 0$, thus have the saturation order and the saturation class. Let $\mathscr{M}(\mathbb{S}^n)$ be the collection of all complex-valued regular Borel measures on $\mathbb{S}^n$, and let $\psi(x)$ be a complex-valued function on the real line. The following function classes will cover the saturation classes of $\oplus^r T^\gamma_{q,t}$ on $\mathscr{L}^p$-space for $p \in [1, +\infty)$ and the continuous function space. Let, see [3, p. 219, Definition 3.2], for $\mathscr{X} = \mathscr{L}^1(\mathbb{S}^n)$,

$$\begin{aligned}\mathscr{H}(\psi; \mathscr{X}) := \{f \in \mathscr{L}^1(\mathbb{S}^n)\colon &\text{there exists } \mu \in \mathscr{M} \text{ such that} \\ &\psi(k) Y_k f = Y_k(\mathrm{d}\mu) \text{ for } k \geqslant 0\};\end{aligned}$$

for $\mathscr{X} = \mathscr{L}^p(\mathbb{S}^n)$, $1 < p < +\infty$,

$$\begin{aligned}\mathscr{H}(\psi; \mathscr{X}) := \{f \in \mathscr{L}^p(\mathbb{S}^n)\colon &\text{there exists } g \in \mathscr{L}^p(\mathbb{S}^n) \\ &\text{such that } \psi(k) Y_k f = Y_k g \text{ for } k \geqslant 0\};\end{aligned} \tag{1.13}$$

for $\mathscr{X} = \mathscr{C}(\mathbb{S}^n)$,

$$\begin{aligned}\mathscr{H}(\psi; \mathscr{X}) := \{f \in \mathscr{C}(\mathbb{S}^n)\colon &\text{there exists } g \in \mathscr{L}^{+\infty}(\mathbb{S}^n) \\ &\text{such that } \psi(k) Y_k f = Y_k g \text{ for } k \geqslant 0\}.\end{aligned} \tag{1.14}$$

Using Berens, Butzer, and Pawelke's method, see [3, Chapter 3], we have the following saturation theorem for $T^\gamma_{q,t}$.

Saturation Theorem *Let $T^\gamma_{q,t}$ be positive and regular. Then $\oplus^r T^\gamma_{q,t}$, $t \geqslant 0$, are saturated with order $\mathscr{O}(t^r)$ and their saturation class is $\mathscr{H}(q^{r\gamma}; \mathscr{X})$.*

If we take $q(x) = x$ and $q(x) = x(x + 2\lambda)$, then $T^\gamma_{q,t}$ becomes the generalized spherical Abel-Poisson sum and the generalized spherical Weierstrass operator, denoted by V^γ_t and W^γ_t, respectively. For $\alpha > 0$, let

$$\omega^\alpha(f, t)_{\mathscr{X}} := \sup\{\|(I - S_\theta)^{\alpha/2} f\|_{\mathscr{X}} : 0 < \theta \leqslant t\}$$

be the modulus of smoothness of degree α for $f \in \mathscr{X}$ and $t \in [0, 2\pi]$ defined in terms of the translation operator on the sphere:

$$S_\theta(f; \boldsymbol{x}) := \frac{1}{|\mathbb{S}^{n-1}| \sin^{n-1}\theta} \int_{\boldsymbol{x}\cdot\boldsymbol{y} = \cos\theta} f(\boldsymbol{y}) \mathrm{d}\widetilde{\sigma}_{\boldsymbol{x}}(\boldsymbol{y}),$$

where $\widetilde{\sigma}_{\boldsymbol{x}}$ is the measure on the subset $\{y \in \mathbb{S}^d \colon \boldsymbol{x} \cdot \boldsymbol{y} = \cos\theta\}$. By (1.12) and using the equivalence between the modulus of smoothness and K-functional, we have the following equivalences for the Booleans $\oplus^r V^\gamma_t$ and $\oplus^r W^\gamma_t$, refer to Theorems 3 and 4 in Section 5.

Main Theorem—Two Examples *For positive integer r and reals $\gamma, \kappa \in (0, 1]$, we have*

$$\|f - \oplus^r V^\gamma_t(f)\|_{\mathscr{X}} \asymp \omega^{r\gamma}(f, t^{1/\gamma})_{\mathscr{X}}, \tag{1.15}$$

$$\|f - \oplus^r W_t^\kappa(f)\|_{\mathscr{X}} \asymp \omega^{2r\kappa}(f, t^{1/(2\kappa)})_{\mathscr{X}}, \tag{1.16}$$

where the constants in (1.15) *depend only on* n, r, *and* γ, *while the constants in* (1.16) *depend only on* n, r, *and* κ.

The equivalence (1.15) reduces to (1.5) when we take $r = 1$ and $\gamma = 1$. Moreover, Theorem 5 in Section 5 shows that $\oplus^r V_t^\gamma$ and $\oplus^r W_t^\kappa$ are both saturated with order t^r and their saturation classes coincide when $\gamma = 2\kappa$.

This paper is organized as follows. Section 2 makes some preparations. Section 3 contains two lemmas about the equivalence between two spherical function classes and that between two K-functionals induced by the multiplier operators. One of these function classes is the saturation class of the positive and regular $T_{q,t}^\gamma$. We leave the proofs of the two lemmas to Section 6. Section 4 proves the Bernstein type inequality for $T_{q,t}^\gamma$ and the equivalence between the approximation error of the Boolean of $T_{q,t}^\gamma$ and the K-functional. Section 5 applies the previous results to the generalized spherical Abel-Poisson sum $\oplus^r V_t^\gamma$ and the generalized Weierstrass operator $\oplus^r W_t^\kappa$.

2 Preliminaries

Denote by c, c_i, or $c(i)$ positive constants, where i is either a positive integer, variable, function, or space on which c depends only. Their values may be different at different occurrences, even within one formula. The notation $a \asymp b$ means that there exists a positive constant c such that

$$c^{-1} b \leqslant a \leqslant cb.$$

By $f(t) = \mathscr{O}_i(t)$, we mean that there exists some constant c_i independent of t such that $|f(t)| \leqslant c_i |t|$, where $f(t)$ is a function with respect to t and we write $f(t) = o(t)$ if $f(t)/t$ tends to zero as $t \to +\infty$ or as $t \to t_0$ where t_0 is a real number. The collection of all positive integers are denoted by $\mathbb{Z}_+$. We denote the generalized binomial coefficient by

$$\binom{r}{k} := \frac{\Gamma(r+1)}{\Gamma(k+1)\Gamma(r-k+1)}.$$

For a sequence $\{a_\ell \colon \ell = 0, 1, \dots\}$, let

$$\overrightarrow{\Delta} a_\ell := a_\ell - a_{\ell+1}$$

be the forward difference of a_ℓ. We will use the method of *summation by parts*: for sequences a_ℓ, b_ℓ, $\ell \geqslant 0$, let

$$B_\ell := \sum_{j=0}^{\ell} b_j, \quad a_{-1} := 0.$$

Then,

$$\sum_{\ell=0}^{n} a_\ell b_\ell = \sum_{\ell=0}^{n} (\overrightarrow{\Delta} a_\ell) B_\ell + a_{n+1} B_n.$$

Let $\mathbb{S}^n$ be the unit sphere of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. For points $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{S}^n$, $\boldsymbol{x} \cdot \boldsymbol{y}$ denotes the inner product in $\mathbb{R}^{n+1}$. Let σ_n be the surface measure on $\mathbb{S}^n$ and denote by σ for convenience if there is no confusion. The volume of $\mathbb{S}^n$ is

$$|\mathbb{S}^n| = \sigma_n(\mathbb{S}^n) = \int_{\mathbb{S}^n} \mathrm{d}\sigma(\boldsymbol{x}) = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

Let A be a statement. We denote by A a.e. if A holds for almost every $\boldsymbol{x} \in \mathbb{S}^n$ with respect to σ.

Function spaces The set $\mathscr{L}^{+\infty}(\mathbb{S}^n)$ forms a Banach space with norm

$$\|f\|_{+\infty} := \|f\|_{\mathscr{L}^{+\infty}(\mathbb{S}^n)} := \operatorname*{ess\,sup}_{x \in \mathbb{S}^n} |f(x)|$$

and the set $\mathscr{L}^p(\mathbb{S}^n)$ forms a Banach space with the norm

$$\|f\|_p := \|f\|_{\mathscr{L}^p(\mathbb{S}^n)} := \left\{ \int_{\mathbb{S}^n} |f(\boldsymbol{x})|^p \mathrm{d}\sigma(\boldsymbol{x}) \right\}^{1/p} < +\infty, \quad 1 \leqslant p < +\infty.$$

The set $\mathscr{C}(\mathbb{S}^n)$ is a Banach space with norm

$$\|f\|_{\mathscr{C}} := \max_{\boldsymbol{x} \in \mathbb{S}^n} |f(\boldsymbol{x})|.$$

The set $\mathscr{M}(\mathbb{S}^n)$ is a Banach space with norm

$$\|\mu\|_{\mathscr{M}} := \int_{\mathbb{S}^n} |\mathrm{d}\mu(\boldsymbol{x})|.$$

We may denote the spaces $\mathscr{L}^p(\mathbb{S}^n)$, $1 \leqslant p \leqslant +\infty$, $\mathscr{C}(\mathbb{S}^n)$, and $\mathscr{M}(\mathbb{S}^n)$ by $\mathscr{L}^p$, $\mathscr{C}$, and $\mathscr{M}$, respectively, for convenience. Let $\mathscr{X}$ be either $\mathscr{L}^p(\mathbb{S}^n)$, $1 \leqslant p < +\infty$, or $\mathscr{C}(\mathbb{S}^n)$. The dual space of $\mathscr{X}$, the collection of all bounded linear functionals on $\mathscr{X}$, is denoted by $\mathscr{X}^*$.

Spherical harmonics Let

$$\Delta := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{n+1}^2}$$

be the Laplace operator on $\mathbb{R}^{n+1}$. A function f on $\mathbb{R}^{n+1}$ is said to be harmonic if $\Delta f = 0$. Denote by $\mathscr{Q}_k^n$ the set of all homogeneous and harmonic polynomials of degree k on $\mathbb{R}^{n+1}$, and by H_k^n the set of the restrictions on $\mathbb{S}^n$ of all functions from $\mathscr{Q}_k^n$. Let Π_k^n be the set of all restrictions on $\mathbb{S}^n$ of polynomials on $\mathbb{R}^{n+1}$. Then Π_k^n is the linear span of H_j^n, $j = 0, 1, \ldots, k$, i.e.,

$$\Pi_k^n = \operatorname{span}\{H_j^n \colon 0 \leqslant j \leqslant k\}.$$

Moreover, $\cup_{k=0}^{+\infty} H_k^n$ is dense in $\mathscr{X}$ and in particular, $\mathscr{L}^2(\mathbb{S}^d)$ is the direct sum of all H_k^n, $k = 0, 1, \ldots$, see, e.g., [16, Theorem 1.1.6].

Convolutions A function $f \in \mathscr{X}$ is said to be a zonal function with $\boldsymbol{x}_0$ on $\mathbb{S}^n$ if for some fixed $\boldsymbol{x}_0 \in \mathbb{S}^n$, $f(\boldsymbol{x}_0 \cdot \boldsymbol{y})$ is a constant when $\boldsymbol{x}_0 \cdot \boldsymbol{y}$ is unchanged. For $1 \leqslant p < +\infty$, the collection of all zonal functions with $\boldsymbol{x}_0$ in $\mathscr{L}^p$ is denoted by $\mathscr{L}_\lambda^p(\mathbb{S}^n, \boldsymbol{x}_0)$ and that in $\mathscr{C}$ by $\mathscr{C}_\lambda(\mathbb{S}^n, \boldsymbol{x}_0)$ (and denote by $\mathscr{L}_\lambda^p$ and $\mathscr{C}_\lambda$, respectively, for convenience, if there is no confusion), where $\lambda = (n-2)/2$. The $\mathscr{L}_\lambda^p(\mathbb{S}^n, \boldsymbol{x}_0)$, $\mathscr{C}_\lambda(\mathbb{S}^n, \boldsymbol{x}_0)$, and $\mathscr{M}_\lambda(\mathbb{S}^n, \boldsymbol{x}_0)$ form Banach spaces, respectively:

$\mathscr{L}_\lambda^p(\mathbb{S}^n, \boldsymbol{x}_0)$ with norm

$$\begin{aligned}\|\varphi\|_{\mathscr{L}_\lambda^p} &:= \left\{ \int_{\mathbb{S}^n} |\varphi(\boldsymbol{x}\cdot\boldsymbol{y})|^p \mathrm{d}\sigma(\boldsymbol{y}) \right\}^{1/p} \\ &= \left\{ |\mathbb{S}^{n-1}| \int_0^\pi |\varphi(\cos\theta)|^p \sin^{2\lambda}\theta \mathrm{d}\theta \right\}^{1/p}; \end{aligned} \tag{2.1}$$

$\mathscr{C}_\lambda(\mathbb{S}^n, \boldsymbol{x}_0)$ with norm

$$\|\varphi\|_{\mathscr{C}_\lambda} := \sup_{0\leqslant\theta\leqslant\pi} |\varphi(\cos\theta)|;$$

$\mathscr{M}_\lambda(\mathbb{S}^n, \boldsymbol{x}_0)$ with norm

$$\|\mu\|_{\mathscr{M}_\lambda} := |\mathbb{S}^{n-1}| \int_0^\pi |\mathrm{d}\mu^*(\theta)|, \tag{2.2}$$

where μ^* is the corresponding function in $\mathscr{M}[0,\pi]$ of the measure $\mu \in \mathbb{S}^n$ (actually, there is a bijection between $\mathscr{M}_\lambda(\mathbb{S}^n, \boldsymbol{x}_0)$ and some subset of $\mathscr{M}[0,\pi]$, see [3,10]).

For $f \in \mathscr{L}^1(\mathbb{S}^n)$ and $\varphi \in \mathscr{L}_\lambda^1(\mathbb{S}^n)$, the convolution of f with the zonal function φ is defined by

$$(f * \varphi)(\boldsymbol{x}) := \int_{\mathbb{S}^n} f(\boldsymbol{y})\varphi(\boldsymbol{x}\cdot\boldsymbol{y})\mathrm{d}\sigma(\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{S}^n. \tag{2.3}$$

The convolution of $\psi \in \mathscr{L}_\lambda^1(\mathbb{S}^n)$ and $\mu \in \mathscr{M}(\mathbb{S}^n)$ is defined by

$$(\psi * \mathrm{d}\mu)(\boldsymbol{x}) := \int_{\mathbb{S}^n} \psi(\boldsymbol{x}\cdot\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}). \tag{2.4}$$

The convolution of $f \in \mathscr{L}^1(\mathbb{S}^n)$ and the zonal measure $\mu \in \mathscr{M}_\lambda(\mathbb{S}^n)$ with $\boldsymbol{x}_0$ is defined by

$$(f * \mathrm{d}\mu)(\boldsymbol{x}) := \int_{\mathbb{S}^n} f(\boldsymbol{y})\mathrm{d}\varphi_{\boldsymbol{x}}\mu(\boldsymbol{y}), \quad \boldsymbol{x} \in \mathbb{S}^n, \tag{2.5}$$

where letting ρ be the rotation such that $\rho\boldsymbol{x} = \boldsymbol{x}_0$, for any measurable subsets $E \subset \mathbb{S}^n$,

$$\varphi_{\boldsymbol{x}}\mu(E) := \mu(\rho E),$$

refer to [3, Chap. 1], [16, Chap. 1], and [10] for details.

Remark 1 The definition of the convolution on the sphere may be found in [3]. In this paper, we follow the definition of convolution in [16]. The convolution we used here differs only a constant from that in [3]. In both definitions, the convolutions admit the following Young's inequalities. For $f \in \mathscr{L}^1(\mathbb{S}^n)$, $\varphi \in \mathscr{L}^1_\lambda(\mathbb{S}^n)$, and $\mu \in \mathscr{M}_\lambda(\mathbb{S}^n)$, we have

$$\|f * \varphi\|_{\mathscr{X}} \leqslant \|\varphi\|_{\mathscr{L}^1_\lambda} \|f\|_{\mathscr{X}}, \tag{2.6a}$$

$$\|\varphi * d\mu\|_1 \leqslant \|\varphi\|_{\mathscr{L}^1_\lambda} \|\mu\|_{\mathscr{M}}. \tag{2.6b}$$

When an operator T on $\mathscr{X}$ is a convolution in the form of (2.3), (2.4), or (2.5), we call $\varphi, \psi \in \mathscr{L}^1_\lambda$ or $\mu \in \mathscr{M}_\lambda$ the *kernel* of T.

Projection For $\nu > -1/2$, let $C_k^{(\nu)}(t)$, $-1 \leqslant t \leqslant 1$, $k = 0, 1, 2, \dots$, be the Gegenbauer polynomial of degree k with ν. The polynomials $C_k^{(\nu)}(t)$, $k \geqslant 0$, form a complete orthogonal basis with respect to the weight $(1-t^2)^{\nu-\frac{1}{2}}$. That is, for $\nu > -1/2$, $\nu \neq 0$, see [15, p. 81],

$$\begin{aligned}\int_{-1}^1 C_k^{(\nu)}(t)C_j^{(\nu)}(t)(1-t^2)^{\nu-1/2}\mathrm{d}t &= \int_0^\pi C_k^{(\nu)}(\cos\theta)C_j^{(\nu)}(\cos\theta)\sin^{2\nu}\theta\mathrm{d}\theta \\ &= \begin{cases} (m(k,\nu))^{-1}, & k = j, \\ 0, & k \neq j, \end{cases}\end{aligned} \tag{2.7}$$

where

$$m(k,\nu) := \frac{2^{2\nu-1}(\Gamma(\nu))^2(k+\nu)\Gamma(k+1)}{\pi\Gamma(k+2\nu)}. \tag{2.8}$$

For $\lambda = (n-2)/2 > 0$, by e.g. [15, p. 171], we have

$$|C_k^{(\lambda)}(t)| = \mathscr{O}_\lambda(k^{2\lambda-1}). \tag{2.9}$$

Let

$$m_k^{(\lambda)} := \frac{\Gamma(\lambda)(k+\lambda)}{2\pi^{\lambda+1}}.$$

Then the projection of $f \in \mathscr{L}^1(\mathbb{S}^n)$ onto H_k^n is defined by, see [3, Chap.1] and [16, Chap.1],

$$Y_k(f;\boldsymbol{x}) := m_k^{(\lambda)} \int_{\mathbb{S}^n} C_k^{(\lambda)}(\boldsymbol{x}\cdot\boldsymbol{y})f(\boldsymbol{y})\mathrm{d}\sigma_n(\boldsymbol{y}) = (f * m_k^{(\lambda)}C_k^{(\lambda)})(\boldsymbol{x}),$$

and for $\mu \in \mathscr{M}(\mathbb{S}^n)$,

$$Y_k(\mathrm{d}\mu;\boldsymbol{x}) := m_k^{(\lambda)} \int_{\mathbb{S}^n} C_k^{(\lambda)}(\boldsymbol{x}\cdot\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) = (m_k^{(\lambda)}C_k^{(\lambda)} * \mathrm{d}\mu)(\boldsymbol{x}).$$

Cesàro mean For real $\alpha > 0$ and $k \in \mathbb{Z}_+$, the Cesàro mean $\tau_k^\alpha(f)$ of $f \in \mathscr{X}$, see, e.g., [16, P. 49], is defined by

$$\tau_k^\alpha(f) := \frac{1}{A_k^\alpha} \sum_{j=0}^{k} A_{k-j}^\alpha Y_j f,$$

where

$$A_k^\alpha := \binom{k+\alpha}{\alpha} = \frac{\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)\Gamma(k+1)}.$$

For $\alpha > \lambda = (n-2)/2$, we have, see, e.g., [16, Theorem 2.3.10],

$$\|\tau_k^\alpha(f)\|_{\mathscr{X}} \leqslant c\|f\|_{\mathscr{X}}, \quad \forall\, k \in \mathbb{Z}_+,\ \forall\, f \in \mathscr{X}, \tag{2.10}$$

where the constant c depends only on n, α, and $\mathscr{X}$.

Multiplier sequences and multiplier operators [3] Let two function spaces $\mathscr{Y}$ and $\mathscr{Z}$ be either $\mathscr{C}(\mathbb{S}^n)$, $\mathscr{L}^p(\mathbb{S}^n)$, $1 \leqslant p < +\infty$, or $\mathscr{M}(\mathbb{S}^n)$. A sequence $\{a_k \in \mathbb{C} \colon k = 0, 1, 2, \dots\}$ is said to be a multiplier sequence from $\mathscr{Y}$ to $\mathscr{Z}$ if for each $f \in \mathscr{Y}$, there exists $g \in \mathscr{Z}$ such that

$$Y_k g = \frac{\lambda}{k+\lambda} a_k Y_k f, \quad k = 0, 1, 2, \dots.$$

Denote by $(\mathscr{Y}, \mathscr{Z})$ the collection of all multiplier sequences from $\mathscr{Y}$ to $\mathscr{Z}$. When $\mathscr{Y} = \mathscr{Z}$, we call $\{a_k\}_{k=0}^{+\infty}$ a multiplier sequence on $\mathscr{Y}$.

Remark 2 By [12, pp. 222–231], for $p \in (1, +\infty)$, we have

$$(\mathscr{M}, \mathscr{M}) = (\mathscr{C}, \mathscr{C}) = (\mathscr{L}^1, \mathscr{L}^1) \subset (\mathscr{L}^p, \mathscr{L}^p) \subset (\mathscr{L}^2, \mathscr{L}^2).$$

For an operator T on $\mathscr{X}$, let

$$\mathscr{D}_1(T) := \{f \in \mathscr{X} : T(f) \in \mathscr{X}\}$$

be the domain of T. We may omit the parenthesis '( )' of $T(f)$ if it does not cause confusion. An operator T on $\mathscr{X}$ is said to be a multiplier operator on $\mathscr{X}$ with the sequence $\{a_k\}_{k=0}^{+\infty}$ if for each $f \in \mathscr{D}_1(T)$,

$$Y_k(Tf) = a_k Y_k(f), \quad k \geqslant 0.$$

If T is a multiplier operator on $\mathscr{X}$ with the sequence a_k, we say that T has an expansion $\sum_{k=0}^{+\infty} a_k Y_k f$ and denote T by

$$Tf \sim \sum_{k=0}^{+\infty} a_k Y_k f.$$

For real $\alpha > 0$, the αth power T^α of T is an operator on $\mathscr{X}$ such that

$$T^\alpha f \sim \sum_{k=0}^{+\infty} (a_k)^\alpha Y_k(f), \quad f \in \mathscr{D}_1(T^\alpha).$$

***K*-functional by multiplier operator** Let $f \in \mathscr{X}$ and $t > 0$. By (1.4), the K-functional induced by the multiplier operator $\mathscr{A}$ with the multiplier sequence $\{a_k\}_{k=0}^{+\infty}$ is given by

$$K_{\mathscr{A}}(f,t)_{\mathscr{X}} := \inf_{g \in \mathscr{D}_1(\mathscr{A})} \{\|f-g\|_{\mathscr{X}} + t\|\mathscr{A}g\|_{\mathscr{X}}\},$$

where

$$\mathscr{D}_1(\mathscr{A}) = \{f \in \mathscr{X} : \text{there exists } g \in \mathscr{X} \text{ such that } a_k Y_k f = Y_k g,\ k = 0,1,2,\dots\} \tag{2.11}$$

is the domain of $\mathscr{A}$. Since the multiplier operator is determined by its multiplier sequence, we also say that $K_{\mathscr{A}}(f,t)_{\mathscr{X}}$ is induced by the multiplier sequence a_k.

In particular, for $\alpha > 0$, the K-functional induced by the $(\alpha/2)$th Laplace-Beltrami operator $(\Delta^*)^{\alpha/2}$ with the multiplier sequence $\{(-k(k+2\lambda))^{\alpha/2}\}_{k=0}^{+\infty}$ is equivalent to the modulus of smoothness:

$$K_{(\Delta^*)^{\alpha/2}}(f,t^\alpha)_{\mathscr{X}} \asymp \omega^\alpha(f,t)_{\mathscr{X}}. \tag{2.12}$$

This was finally proved by Riemenschneider and Wang [14].

We may also define the K-functional induced by infinitesimal generator as follows. For a semigroup T_t of class $(\mathscr{C}_0)$ and integer $r \geqslant 1$, recall that $\mathscr{A}^r$ is the rth power of its infinitesimal generator $\mathscr{A}$, see (1.9b).

***K*-functional by infinitesimal generator** [6, Section 3.4] For $f \in \mathscr{X}$, the K-functional induced by the rth power $\mathscr{A}^r$ is defined by

$$K^*_{\mathscr{A}^r}(f,t)_{\mathscr{X}} := \inf_{g \in \mathscr{D}(\mathscr{A}^r)} \{\|f-g\|_{\mathscr{X}} + t\|\mathscr{A}^r g\|_{\mathscr{X}}\}, \tag{2.13}$$

where $\mathscr{D}(\mathscr{A}^r)$ is defined by (1.9a).

Remark 3 For integer $r \geqslant 2$, when the infinitesimal generator $\mathscr{A}$ of a semigroup is a multiplier operator, so will $\mathscr{A}^r$. As a multiplier operator, the $\mathscr{A}^r$ for positive integer r induces the K-functional $K_{\mathscr{A}^r}(f,t)_{\mathscr{X}}$. We note that $K^*_{\mathscr{A}^r}(f,t)_{\mathscr{X}}$ usually does not equal $K_{\mathscr{A}^r}(f,t)_{\mathscr{X}}$. If the operators T_t that form a semigroup of class $(\mathscr{C}_0)$ admit the expansions of $\sum_{k=0}^{+\infty} \mathrm{e}^{a(k)t} Y_k f$ for $f \in \mathscr{X}$, we have $\mathscr{D}(\mathscr{A}^r) \subset \mathscr{D}_1(\mathscr{A}^r)$, see Lemma 2 below. In this case,

$$K_{\mathscr{A}^r}(f,t)_{\mathscr{X}} \leqslant K^*_{\mathscr{A}^r}(f,t)_{\mathscr{X}},$$

see Remark 8 below.

3 Function classes and *K*-functionals

We study in this section the classes of functions induced by the multiplier sequences and the K-functionals induced by the multiplier operators. We prove the equivalence between two classes of functions and that between two K-functionals.

The following lemma plays an important role in the proof of the equivalence between the K-functionals. For function $\psi(x)$ from $\mathbb{R}$ to $\mathbb{C}$, let

$$\mathscr{H}_1(\psi;\mathscr{X}) := \{f \in \mathscr{X}:\ \text{there exists } g \in \mathscr{X} \text{ such that } \psi(k) Y_k f = Y_k g \text{ for } k \geqslant 0\}$$

be a set of functions on $\mathscr{X}$, determined by the sequence $\psi(k)$. We note that

$$\mathscr{H}(\psi;\mathscr{L}^p(\mathbb{S}^n)) = \mathscr{H}_1(\psi;\mathscr{L}^p(\mathbb{S}^n)), \quad 1 < p < +\infty,$$

see (1.14). For integer $k \geqslant 0$, let $\mathscr{C}^k[0,+\infty)$ be the set of all k times continuously differentiable real functions on $[0,+\infty)$. We have the following result.

Lemma 1 *Let ψ_0 and φ_0 be two complex-valued function on $[0,+\infty)$, and let ψ and φ be two real functions such that*

$$\psi(x) = \mathrm{e}^{\mathrm{i}v_1\pi}\psi_0(x), \quad \varphi(x) = \mathrm{e}^{\mathrm{i}v_2\pi}\varphi_0(x)$$

for some reals v_1 and v_2, and

$$0 < \lim_{x\to+\infty}\frac{\psi(x)}{\varphi(x)} = c_0 < +\infty, \quad \psi(0) = \varphi(0) = 0.$$

Let

$$g(t) := \begin{cases} \dfrac{\psi(t^{-1})}{\varphi(t^{-1})}, & 0 < t < +\infty, \\ c_0, & t = 0, \end{cases}$$

and let

$$g^{(i)}(0) := \lim_{t\to 0+}\frac{g^{(i-1)}(t) - g^{(i-1)}(0)}{t}, \quad 1 \leqslant i \leqslant 2\lambda + 2.$$

If functions g and $1/g$ are both in $\mathscr{C}^{2\lambda+2}[0,+\infty)$, then for real s,

$$\mathscr{H}_1((\psi_0)^s;\mathscr{X}) = \mathscr{H}_1((\varphi_0)^s;\mathscr{X}).$$

We leave the proof of Lemma 1 to Section 6.

Remark 4 For function $a(x)$ on $\mathbb{R}$, let $\mathscr{A}$ be a multiplier operator with a multiplier sequence $a(k)$. Then the domain $\mathscr{D}_1(\mathscr{A}^\alpha)$, see (2.11), of $\mathscr{A}^\alpha$ is $\mathscr{H}_1(a^\alpha;\mathscr{X})$.

Remark 5 Let φ_0 and ψ_0 be given by Lemma 1. We may analogously prove

$$\mathscr{H}((\varphi_0)^s;\mathscr{X}) = \mathscr{H}((\psi_0)^s;\mathscr{X})$$

by

$$(\mathscr{M},\mathscr{M}) = (\mathscr{C},\mathscr{C}) \subset (\mathscr{L}^p,\mathscr{L}^p), \quad 1 \leqslant p \leqslant +\infty,$$

see Remark 2.

For two polynomials $a(x)$ and $b(x)$ that satisfy the assumptions of Lemma 1, we may use Lemma 1 and the method of [7] to prove the equivalence between the K-functionals induced by the multiplier sequences $a(k)$ and $b(k)$ as follows. We leave the proof of the lemma to Section 6.

Lemma 2 *Let $a(x)$ and $b(x)$ be two polynomials with the same degree, and let real $\alpha > 0$. Let the operators $\mathscr{A}$ and $\mathscr{B}$ on $\mathscr{X}$ be with multiplier sequences $\{a(k)\}_{k=0}^{+\infty}$ and $\{b(k)\}_{k=0}^{+\infty}$, respectively. If $a(x)$ and $b(x)$ satisfy the assumptions of Lemma 1 and neither $a(x)$ nor $b(x)$ have positive zero points, then*

$$\mathscr{D}_1(\mathscr{A}^\alpha) = \mathscr{D}_1(\mathscr{B}^\alpha)$$

and

$$K_{\mathscr{A}^\alpha}(f,t)_{\mathscr{X}} \asymp K_{\mathscr{B}^\alpha}(f,t)_{\mathscr{X}}, \quad t > 0, \ f \in \mathscr{D}_1(\mathscr{A}^\alpha),$$

where the constants in the inequalities depend only on $a(\cdot)$, $b(\cdot)$, α, and $\mathscr{X}$.

4 Approximation for semigroups of class ($\mathscr{C}_0$) on spheres

Recall that $T^\gamma_{q,t}$ is the exponential type multiplier operator with a polynomial q and $0 < \gamma \leqslant 1$, given in the introduction. In this section, we prove that if $T^\gamma_{q,t}$ are regular and positive, then $T^\gamma_{q,t}$ form a contraction semigroup of class ($\mathscr{C}_0$). Moreover, the semigroup $T^\gamma_{q,t}$ admits the Bernstein type inequality, see (1.10). From this inequality, using the method of [9], we show that the approximation error of its rth Boolean $\oplus^r T^\gamma_{q,t}$ is equivalent to the K-functional induced by the multiplier sequence $(q(k))^{r\gamma}$.

Bochner integral Let (X, μ) be a measure space, and let B be a Banach space with norm $\|\cdot\|_B$. For a measurable vector-valued function $f\colon X \to B$, one may define the Bochner integral of f as follows:

$$\int_X f(t)\mathrm{d}\mu(t). \tag{4.1}$$

Let E be μ-measurable. A function $f\colon E \to B$ is Bochner integrable on $E \subset X$ if and only if $\|f(t)\|$ is Lebesgue measurable with respect to t and

$$\int_E \|f\|\mathrm{d}\mu(t) < +\infty.$$

Moreover, for a Bochner integrable function f and a μ-measurable set $E \subset X$, we have

$$\Big\| \int_E f(t)\mathrm{d}\mu(t)\Big\|_B \leqslant \int_E \|f(t)\|_B\mathrm{d}\mu(t). \tag{4.2}$$

Let T be a closed linear operator on B, and let $f\colon X \to B$ is Bochner integrable. If Tf is also Bochner integrable, then T commutes with the Bochner integral, i.e., for every μ-measurable set $E \subset X$,

$$T\biggl(\int_E f(t)\mathrm{d}\mu(t)\biggr) = \int_E T(f(t))\mathrm{d}\mu(t). \tag{4.3}$$

Remark 6 Let I be an interval of the real line. The set I with the usual Lebesgue measure forms a measure space. A vector-valued function $h\colon I \to B$ is said to be strongly continuous if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for $|t_1 - t_2| < \delta$, $t_1, t_2 \in I$, we have

$$\|h(t_1) - h(t_2)\|_B < \varepsilon.$$

Since the continuity of $\|h(t)\|$ implies that $\|h(t)\|$ is Lebesgue measurable, we know that any strongly continuous vector-valued function $h\colon I \to B$ is Bochner integrable.

We need the following lemma that shows an integral representation of the semigroup of the exponential type multiplier operators. Let $\{T_t\colon 0 \leqslant t < +\infty\}$ be a semigroup of class $(\mathscr{C}_0)$. For each $t \geqslant 0$, T_t is a multiplier operator with a sequence of multipliers in the form of $\mathrm{e}^{a_k t}$ for some real sequence $\{a_k\}_{k=0}^{+\infty}$.

Lemma 3 *Let $\{T_t\colon 0 \leqslant t < +\infty\}$ be a semigroup of class $(\mathscr{C}_0)$ on $\mathscr{X}$ and also a semigroup of the exponential type multiplier operators with multipliers $a_t(k) = \mathrm{e}^{a_k t}$. Then for $r \in \mathbb{Z}_+$, we have*

$$\mathscr{D}(\mathscr{A}^r) \subset \mathscr{D}_1(\mathscr{A}^r) \tag{4.4}$$

and

$$\mathscr{A}^r f \sim \sum_{k=0}^{+\infty} (a_k)^r Y_k f,$$

where $\mathscr{D}(\mathscr{A}^r)$ and $\mathscr{D}_1(\mathscr{A}^r)$ are given by (1.9) *and* (2.11), *respectively. In particular, for $r = 1$, $\mathscr{D}(\mathscr{A}) = \mathscr{D}_1(\mathscr{A})$. Moreover, for $f \in \mathscr{D}(\mathscr{A}^r)$ and $g \in \mathscr{X}$ such that*

$$(a_k)^r Y_k f = Y_k g, \quad k \geqslant 0,$$

we have

$$(T_t - I)^r f = \int_0^t \cdots \int_0^t T_{u_1 + \cdots + u_r} g \mathrm{d}u_1 \cdots \mathrm{d}u_r, \quad \text{a.e.} \tag{4.5}$$

We leave the proof of Lemma 3 to Section 6.

Remark 7 In Lemma 3, the sequence a_k does not have to be a polynomial. Neither needs the operator positive.

The right-hand side of (4.5) is the Bochner integral of a vector-valued function $h\colon [0, +\infty)^r \to \mathscr{X}$ over the subset $[0, t]^r$.

Remark 8 With the same assumptions of Lemma 3, by (4.4), we have

$$K_{\mathscr{A}^r}(f,t)_{\mathscr{X}} \leqslant K^*_{\mathscr{A}^r}(f,t)_{\mathscr{X}}, \quad f \in \mathscr{X},\ t > 0. \tag{4.6}$$

We turn to study the exponential type multiplier operator $T^{\gamma}_{q,t}$ on $\mathscr{X}$ which for $t > 0$ has the expansion of (1.6). Recall that $T^{\gamma}_{q,t}$ is *regular* if the coefficient of the first term of $q(x)$ and the degree of $q(x)$ are both positive and $q(0) = 0$.

Remark 9 The multiplier operator $T^{\gamma}_{q,t}$ has a kernel $\varphi^{\gamma}_{q,t}$. For $f \in \mathscr{X}$,

$$T^{\gamma}_{q,t} f = f * \varphi^{\gamma}_{q,t}, \tag{4.7}$$

where $\varphi^{\gamma}_{q,t}$ is given by

$$\varphi^{\gamma}_{q,t}(\cos\theta) = \frac{1}{|\mathbb{S}^n|}\sum_{k=0}^{+\infty} \mathrm{e}^{-(q(k))^{\gamma} t}\,\frac{k+\lambda}{\lambda}\,C^{(\lambda)}_k(\cos\theta).$$

Then, for $0 < \gamma \leqslant 1$ and $t \geqslant 0$, $\varphi^{\gamma}_{q,t}(\cos\theta) \in \mathscr{L}^1_{\lambda}$. For $r \in \mathbb{Z}_+$, let $(\mathscr{A}^{\gamma}_q)^r$ be an operator such that

$$(\mathscr{A}^{\gamma}_q)^r f \sim \sum_{k=0}^{+\infty} (-(q(k))^{\gamma})^r Y_k f, \quad (\mathscr{A}^{\gamma}_q)^r f \in \mathscr{X}. \tag{4.8}$$

The following theorem shows that the regular and positive operators $T^{\gamma}_{q,t}$, $t \geqslant 0$, form a semigroup of class $(\mathscr{C}_0)$. The multiplier operator $\mathscr{A}^{\gamma}_q$ is actually the infinitesimal generator of the semigroup and it admits the following Bernstein type inequality.

Theorem 1 *Let* $\{T^{\gamma}_{q,t} \colon 0 \leqslant t < +\infty\}$ *defined by* (1.6) *be a set of regular and positive exponential type multiplier operators on* $\mathscr{X}$. *Then* $\{T^{\gamma}_{q,t} \colon 0 \leqslant t < +\infty\}$ *forms a strongly continuous semigroup of contraction operators of class* $(\mathscr{C}_0)$, *and for* $t > 0$, $f \in \mathscr{X}$, $T^{\gamma}_{q,t} f \in \mathscr{D}(\mathscr{A}^{\gamma}_q)$. *Moreover,*

$$\|\mathscr{A}^{\gamma}_q T^{\gamma}_{q,t} f\|_{\mathscr{X}} \leqslant \frac{c}{t}\,\|f\|_{\mathscr{X}}, \tag{4.9}$$

where c *is a constant depending only upon* n, γ, $q(\cdot)$, *and* $\mathscr{X}$.

Proof For $t_1, t_2 > 0$ and $f \in \mathscr{X}$, we have

$$(T^{\gamma}_{q,t_1} T^{\gamma}_{q,t_2}) f = T^{\gamma}_{q,t_1+t_2} f, \tag{4.10}$$

and by (2.1), the positivity of $\varphi^{\gamma}_{q,t}(\cos\theta)$, and (2.7),

$$\|\varphi^{\gamma}_{q,t}(\cos(\cdot))\|_{\mathscr{L}^1_{\lambda}} = |\mathbb{S}^{n-1}| \int_0^{\pi} \varphi^{\gamma}_{q,t}(\cos\theta)\sin^{2\lambda}\theta \mathrm{d}\theta = 1.$$

By Young's inequality (2.6a) and (4.7), we have

$$\|T^{\gamma}_{q,t}f\|_{\mathscr{X}} \leqslant \|\varphi^{\gamma}_{q,t}(\cos(\cdot))\|_{\mathscr{L}^1_\lambda}\,\|f\|_{\mathscr{X}} = \|f\|_{\mathscr{X}}, \tag{4.11}$$

and also, for $f \in \mathscr{X}$,

$$\lim_{t\to 0+}\|T^{\gamma}_{q,t}f - f\|_{\mathscr{X}} = 0, \tag{4.12}$$

which is by (4.11), the contraction of $T^{\gamma}_{q,t}$, and the Banach-Steinhaus theorem, as well as the fact that the collection of all spherical polynomials is dense in $\mathscr{X}$. By Lemma 3 and Remark 9, for any $f \in \mathscr{X}$ and $t > 0$, we may verify that

$$T^{\gamma}_{q,t}f \in \mathscr{D}_1(\mathscr{A}^{\gamma}_q) = \mathscr{D}(\mathscr{A}^{\gamma}_q),$$

and since the projection Y_k commutes with $T^{\gamma}_{q,t}$, we have

$$\mathscr{A}^{\gamma}_q T^{\gamma}_{q,t}f = -\sum_{k=0}^{+\infty}(q(k))^{\gamma}Y_k(T^{\gamma}_{q,t}f) = -\sum_{k=0}^{+\infty}(q(k))^{\gamma}\mathrm{e}^{-(q(k))^{\gamma}t}Y_kf, \quad \text{a.e.} \tag{4.13}$$

Then, by (1.7), (4.10)–(4.12), we have $T^{\gamma}_{q,t}$, $t \geqslant 0$, form a strongly continuous contraction semigroup of class $(\mathscr{C}_0)$.

Now, we are going to prove (4.9). Let d be the degree of $q(x)$. There exist constants c and c' such that

$$cx^{\beta} \leqslant (q(x))^{\gamma} \leqslant c'x^{\beta}, \quad 0 < x < +\infty,\ \beta = \gamma d. \tag{4.14}$$

Then,

$$\begin{aligned}\|\mathscr{A}^{\gamma}_q T^{\gamma}_{q,t}f\|_{\mathscr{X}} &= \Big\|\sum_{k=1}^{+\infty}\delta^{l+1}((q(k))^{\gamma}\mathrm{e}^{-(q(k))^{\gamma}t})A^l_k\sigma^l_k(f)\Big\|_{\mathscr{X}}\\ &\leqslant c\sum_{k=1}^{+\infty}|\delta^{l+1}((q(k))^{\gamma}\mathrm{e}^{-(q(k))^{\gamma}t})k^l|\,\|f\|_{\mathscr{X}},\end{aligned} \tag{4.15}$$

where the first equality uses $q(0) = 0$ and summation by parts $(l+1)$ times, and l is a positive integer larger than $\lambda = (n-2)/2$.

We need to estimate

$$\Big|\sum_{k=1}^{+\infty}\delta^{l+1}((q(k))^{\gamma}\mathrm{e}^{-(q(k))^{\gamma}t})k^l\Big|.$$

By induction,

$$\begin{aligned}&\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^l((q(x))^{\gamma}\mathrm{e}^{-(q(x))^{\gamma}t})\\ &\quad= \sum_{i=0}^{l}\mathrm{e}^{-(q(x))^{\gamma}t}\sum_{v=1}^{N'_{l-i}}\sum_{j=1}^{N_i}t^{s_{iv}}(q(x))^{(s_{iv}+1)\gamma-(m_{iv}+r_{ij})}Q^{d(m_{iv}+r_{ij})-(n_{iv}+i)}_{ivj}(x),\end{aligned}$$

where

$$0 \leqslant r_{ij} \leqslant i, \quad 0 \leqslant s_{iv}, m_{iv} \leqslant l-i, \quad n_{iv} \geqslant l-i,$$

N_i, N_i' are all positive integers, Q^d_{ivj}, $d = 0, 1, 2, \ldots$, are polynomials with degree d, and

$$d(m_{iv} + r_{ij}) - (n_{iv} + i) \geqslant 0.$$

Thus, by (4.14),

$$n_{iv} \geqslant l-i, \quad s_{iv} \leqslant l-i,$$

and for $x \geqslant 1$, we have

$$\begin{aligned}
\Big|\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^l ((q(x))^\gamma \mathrm{e}^{-(q(x))^\gamma t})\Big| &\leqslant \sum_{i=0}^{l} \mathrm{e}^{-(q(x))^\gamma t} \sum_{v=1}^{N'_{l-i}} \sum_{j=1}^{N_i} \ell_{ivj} t^{s_{iv}} x^{(s_{iv}+1)\beta-(n_{iv}+i)} \\
&\leqslant \sum_{i=0}^{l} \mathrm{e}^{-cx^\beta t} \sum_{v=1}^{N'_{l-i}} \ell_{iv} t^{s_{iv}} x^{(s_{iv}+1)\beta - l} \\
&= \sum_{i=0}^{l} \ell_i t^i x^{(i+1)\beta - l} \mathrm{e}^{-cx^\beta t},
\end{aligned}$$

where we rewrite the sum in the last equality. By the relation between the finite difference of the sequence $q(k)$ and the derivatives of function $q(x)$, we have

$$\begin{aligned}
&|\delta^{l+1}((q(k))^\gamma \mathrm{e}^{-(q(k))^\gamma t})| \\
&= \Big| \int_0^1 \cdots \int_0^1 \Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^{l+1} ((q(x))^\gamma \mathrm{e}^{-(q(x))^\gamma t})\big|_{x=k+u_1+\cdots+u_{l+1}} \mathrm{d}u_1 \cdots \mathrm{d}u_{l+1} \Big| \\
&\leqslant \sum_{i=0}^{l+1} \ell'_i t^i k^{(i+1)\beta-(l+1)} \mathrm{e}^{-ck^\beta t},
\end{aligned}$$

where ℓ'_i, $i = 0, 1, \ldots, l+1$, are positive constants depending only upon $q(x)$, γ, i, and l. Hence,

$$\Big| \sum_{k=1}^{+\infty} \delta^{l+1}((q(k))^\gamma \mathrm{e}^{-(q(k))^\gamma t}) k^l \Big| \leqslant \sum_{i=0}^{l+1} \ell'_i t^i \sum_{k=1}^{+\infty} k^{(i+1)\beta-1} \mathrm{e}^{-ck^\beta t}. \tag{4.16}$$

Now, we consider the derivative of function $g(x) = x^{(i+1)\beta-1}\mathrm{e}^{-cx^\beta t}$, $i = 0, 1, \ldots, m$:

$$\frac{\mathrm{d}}{\mathrm{d}x} g(x) = (((i+1)\beta - 1) + (-c\beta t)x^\beta) x^{(i+1)\beta-2} \mathrm{e}^{-cx^\beta t}.$$

Then there exists integer $k_i \geqslant 0$ (may depend on t) such that $g(x)$ is decreasing in $[1, k_i]$ and increasing in $(k_i, +\infty)$. By (4.16), summing up over $1 \leqslant k \leqslant k_i$

and $k > k_i$ respectively, and using the above monotonic properties of $g(x)$, we have

$$\begin{aligned}
&\Big|\sum_{k=1}^{+\infty}\delta^{l+1}(k^{\beta}\mathrm{e}^{-ck^{\beta}t})k^{l}\Big| \\
&\quad\leqslant \sum_{i=0}^{l+1}\ell_i' t^i \sum_{k=1}^{+\infty} k^{(i+1)\beta-1}\mathrm{e}^{-ck^{\beta}t} \\
&\quad\leqslant \sum_{i=0}^{l+1}\ell_i' t^i \Big(\sum_{k=1}^{k_i}\int_k^{k+1}\mathrm{e}^{-cx^{\beta}t}x^{(i+1)\beta-1}\mathrm{d}x + \sum_{k=k_i+1}^{+\infty}\int_{k-1}^{k}\mathrm{e}^{-cx^{\beta}t}x^{(i+1)\beta-1}\mathrm{d}x\Big) \\
&\quad\leqslant \sum_{i=0}^{l+1}(2\ell_i' t^i)\int_0^{+\infty}\mathrm{e}^{-cx^{\beta}t}x^{(i+1)\beta-1}\mathrm{d}x \\
&\quad= \sum_{i=0}^{l+1}(2\ell_i' t^i)(i!\,c^{-(i+1)}t^{-(i+1)}\beta^{-1}) \\
&\quad= \frac{c_1}{t},
\end{aligned}$$

where

$$c_1 = \Big(\sum_{i=0}^{l+1}2\ell_i' c^{-(i+1)}i!\Big)\beta^{-1}.$$

Using (4.15), we have

$$\|\mathscr{A}_q^{\gamma}T_{q,t}^{\gamma}f\|_{\mathscr{X}} \leqslant \frac{c_2}{t}\,\|f\|_{\mathscr{X}},$$

where the constant c_2 depends only on q, γ, n, and $\mathscr{X}$. This completes the proof. □

With the Bernstein inequality (4.9), we may prove the equivalence between approximation error of the semigroup $T_{q,t}^{\gamma}$ and the K-functional induced by its infinitesimal generator $\mathscr{A}_q^{\gamma}$, as follows.

Theorem 2 *Let $\{T_{q,t}^{\gamma}\colon 0 \leqslant t < +\infty\}$ defined by (1.6) be a set of regular and positive exponential type multiplier operators on $\mathscr{X}$. Then, for $r \in \mathbb{Z}_+$,*

$$\|\oplus^r T_{q,t}^{\gamma}f - f\|_{\mathscr{X}} \asymp K_{(\mathscr{A}_q^{\gamma})^r}(f, t^r)_{\mathscr{X}}, \quad \forall\ f \in \mathscr{X}, \tag{4.17}$$

where $\mathscr{A}_q^{\gamma}$ is given by (4.8), where the constants depend only on n, r, q, γ, and $\mathscr{X}$.

Moreover, $\{\oplus^r T_{q,t}^{\gamma}\colon 0 \leqslant t < +\infty\}$ are saturated with order $\mathscr{O}(t^r)$ and their saturation class is $\mathscr{H}(q^{r\gamma};\mathscr{X})$.

We need to make some preparations before the proof of Theorem 2. We use the following result of Ditzian and Ivanov to prove (4.17).

Lemma 4 [9, Theorem 5.1] *Let $\{T_t\colon 0 \leqslant t < +\infty\}$ be a contraction semigroup of class ($\mathscr{C}_0$) on $\mathscr{X}$. The operator $\mathscr{A}$ is the infinitesimal generator of T_t. If there exists some constant c independent of t and f such that*

$$t\|\mathscr{A}T_t f\|_{\mathscr{X}} \leqslant c\|f\|_{\mathscr{X}},$$

then, for any $r \in \mathbb{Z}_+$,

$$\| \oplus^r T_t f - f\|_{\mathscr{X}} \asymp K^*_{\mathscr{A}^r}(f, t^r)_{\mathscr{X}},$$

where the $K^*_{\mathscr{A}^r}(f,\cdot)_{\mathscr{X}}$ is given by (2.13) *and the constants in the inequalities are independent of t and f.*

Gegenbauer coefficients Let $\varphi(\boldsymbol{x}\cdot\boldsymbol{y})$ be a zonal function on $\mathbb{S}^n$. The Gegenbauer coefficients $\widehat{\varphi}(k)$, $k = 0, 1, 2, \dots$, are defined by

$$\widehat{\varphi}(k) := |\mathbb{S}^n| m(k,\lambda) \int_0^\pi \varphi(\cos\theta) C_k^{(\lambda)}(\cos\theta) \sin^{n-1}\theta \mathrm{d}\theta.$$

Since $\varphi(\boldsymbol{x}\cdot\boldsymbol{y})$ coincides with the kernel $\varphi(\cos\theta)$, we also say that $\widehat{\varphi}(k)$ are the Gegenbauer coefficients of the kernel $\varphi(\cos\theta)$.

The saturation property of the multiplier operator is determined by the Gegenbauer coefficients of the kernel of the operator.

Lemma 5 [3, Theorem 3.1] *Let $\{T_t\colon t \geqslant 0\}$ be a set of multiplier operators on $\mathscr{X}$ with kernels $\phi_t(\cos\theta)$, $0 \leqslant \theta \leqslant \pi$. If*

$$\int_{\mathbb{S}^n} \phi_t(\boldsymbol{x}\cdot\boldsymbol{y}) \mathrm{d}\sigma(\boldsymbol{y}) = 1,$$

$$\|T_t f\|_{\mathscr{X}} \leqslant c\|f\|_{\mathscr{X}},$$

and there exist a function $\psi(t)$, $t \geqslant 0$, converging to zero as $t \to 0$ and a sequence $q(k)$ such that

$$\lim_{t\to 0} \frac{\frac{\lambda}{n+\lambda}\widehat{\phi}_t(k) - 1}{\psi(t)} = q(k), \quad k \geqslant 0.$$

Then we have $f \in \mathscr{H}(q;\mathscr{X})$ if $\|T_t f - f\|_{\mathscr{X}} = \mathscr{O}(\psi(t))$ and f is a constant if $\|T_t f - f\|_{\mathscr{X}} = o(\psi(t))$.

Proof of Theorem 2 By definition, we have

$$\| \oplus^r T^\gamma_{q,t} f - f\|_{\mathscr{X}} \leqslant c_{n,r,q,\gamma,\mathscr{X}} K_{\mathscr{A}_q^\gamma}(f, t^r)_{\mathscr{X}}.$$

On the other hand, by (4.9), Lemma 4, and (4.6), we have

$$\| \oplus^r T^\gamma_{q,t} f - f\|_{\mathscr{X}} \geqslant c_{n,r,q,\gamma,\mathscr{X}} K^*_{\mathscr{A}_q^\gamma}(f, t^r)_{\mathscr{X}} \geqslant c_{n,r,q,\gamma,\mathscr{X}} K_{\mathscr{A}_q^\gamma}(f, t^r)_{\mathscr{X}}.$$

This proves (4.17).

For the saturation property of $\{\oplus^r T^\gamma_{q,t}\colon 0 \leqslant t < +\infty\}$, we let $\mathscr{A}^\gamma_q$ be the infinitesimal generator of $T^\gamma_{q,t}$, and let $\{\widehat{\varphi}^\gamma_{r,q,t}(k)\}_{k=0}^{+\infty}$ be the Gegenbauer coefficients of the kernel $\varphi^\gamma_{r,q,t}(\cos\theta)$ of $\oplus^r T^\gamma_{q,t}$. Then

$$\varphi^\gamma_{r,q,t}(\cos\theta) = \frac{1}{|\mathbb{S}^n|}\sum_{k=0}^{+\infty}(1-(1-\mathrm{e}^{-(q(k))^\gamma t})^r)\frac{k+\lambda}{\lambda}C_k^{(\lambda)}(\cos\theta)$$

is in $\mathscr{C}_\lambda(\mathbb{S}^n)$, and

$$\widehat{\varphi}^\gamma_{r,q,t}(k) = \frac{k+\lambda}{\lambda}(1-(1-\mathrm{e}^{-(q(k))^\gamma t})^r), \quad k \geqslant 0.$$

Thus, we have

$$\lim_{t\to 0+}\frac{\frac{\lambda}{k+\lambda}\widehat{\varphi}^\gamma_{r,q,t}(k)-1}{t^r} = -(q(k))^{r\gamma}, \quad k = 0,1,2,\ldots. \tag{4.18}$$

In addition,

$$|\mathbb{S}^{n-1}|\int_0^\pi \varphi^\gamma_{r,q,t}(\cos\theta)\sin^{2\lambda}\theta \mathrm{d}\theta = 1 \tag{4.19}$$

and

$$\|\oplus^r T^\gamma_{q,t}f\|_{\mathscr{X}} \leqslant 2^r\|f\|_{\mathscr{X}}. \tag{4.20}$$

Using Lemma 5 and by (4.18)–(4.20), we have $f \in \mathscr{H}(q^{r\gamma};\mathscr{X})$ if

$$\|\oplus^r T^\gamma_{q,t}f - f\|_{\mathscr{X}} = \mathscr{O}(t^r)$$

and f is a constant if

$$\|\oplus^r T^\gamma_{q,t}f - f\|_{\mathscr{X}} = o(t^r).$$

On the other hand, suppose $f \in \mathscr{H}(q^{r\gamma};\mathscr{X})$. In the following, we prove

$$\|\oplus^r T^\gamma_{q,t}f - f\|_{\mathscr{X}} = \mathscr{O}(t^r).$$

We prove it only for the case $\mathscr{X} = \mathscr{L}^1(\mathbb{S}^n)$. The proofs for $\mathscr{L}^1(\mathbb{S}^n)$ and $\mathscr{C}(\mathbb{S}^n)$ are similar. For $f \in \mathscr{H}(q^{r\gamma};\mathscr{X})$, there exists $g \in \mathscr{L}^p(\mathbb{S}^n)$ such that

$$(-(q(k))^\gamma)^r Y_k f = Y_k g, \quad k \geqslant 0.$$

By Lemma 3 and (4.2), we have

$$\begin{aligned}\|\oplus^r T^\gamma_{q,t}f - f\|_p &= \|(T^\gamma_{q,t}-I)^r f\|_p\\ &= \Big\|\int_0^t\cdots\int_0^t T^\gamma_{q,u_1+\cdots+u_r}g\mathrm{d}u_1\cdots\mathrm{d}u_r\Big\|_p\\ &\leqslant \|g\|_p t^r\\ &= \mathscr{O}(t^r).\end{aligned}$$

In the remaining of the proof, we use the method from [3]. Since $f \in \mathscr{H}(p^{r\gamma}; \mathscr{L}^1(\mathbb{S}^n))$, there exists $\mu \in \mathscr{M}(\mathbb{S}^n)$ such that

$$(-(q(k))^{\gamma})^r Y_k f = Y_k(\mathrm{d}\mu), \quad k \geqslant 0. \tag{4.21}$$

The convolution

$$(\varphi_{q,t}^{\gamma} * \mathrm{d}\mu)(\boldsymbol{x}) := \int_{\mathbb{S}^n} \varphi_{q,t}^{\gamma}(\boldsymbol{x}\cdot\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y})$$

is in $\mathscr{L}^1(\mathbb{S}^n)$. By Young's inequality (2.6b), we have

$$\|\varphi_{q,t}^{\gamma} * \mathrm{d}\mu\|_1 \leqslant \|\varphi_{q,t}^{\gamma}\|_{\mathscr{L}_\lambda^1} \|\mu\|_{\mathscr{M}} = \|\mu\|_{\mathscr{M}}. \tag{4.22}$$

For given $\mu \in \mathscr{M}(\mathbb{S}^n)$, $h(t) = \varphi_{q,t}^{\gamma} * \mathrm{d}\mu$ defines a vector-valued function from $(0, +\infty)$ to $\mathscr{L}^1(\mathbb{S}^n)$ and we may verify that for any $\varepsilon > 0$,

$$h(t) = \varphi_{q,t-\varepsilon}^{\gamma} * (\varphi_{q,\varepsilon}^{\gamma} * \mathrm{d}\mu) = \varphi_{q,t-\varepsilon}^{\gamma} * h(\varepsilon) = T_{q,t-\varepsilon}^{\gamma} h(\varepsilon).$$

Then for $0 < \varepsilon \leqslant t_2 < t_1 < +\infty$, by the contraction of $T_{q,t}^{\gamma}$, we have

$$\begin{aligned}
\|h(t_1) - h(t_2)\|_1 &= \|T_{q,t_1-\varepsilon}^{\gamma} h(\varepsilon) - T_{q,t_2-\varepsilon}^{\gamma} h(\varepsilon)\|_1 \\
&= \|T_{q,t_2-\varepsilon}^{\gamma}(T_{q,t_1-t_2}^{\gamma} h(\varepsilon) - h(\varepsilon))\|_1 \\
&\leqslant \|T_{q,t_1-t_2}^{\gamma} h(\varepsilon) - h(\varepsilon)\|_1 \\
&\to 0, \quad t_1 \to t_2,
\end{aligned}$$

where we used (4.12). Therefore, $h(t)$ is strongly continuous in $[\varepsilon, +\infty)$ for $\varepsilon > 0$.

By (4.22), we have

$$\int_{\varepsilon}^{t} \|h(\tau)\|_1 \mathrm{d}\tau \leqslant \int_{\varepsilon}^{t} \|\mu\|_{\mathscr{M}} \mathrm{d}\tau < \|\mu\|_{\mathscr{M}} t, \quad \forall\, \varepsilon > 0.$$

It follows that $h(t)$ is Bochner integrable on $(0, t]$, see Remark 6. For $k = 0, 1, 2, \ldots$, by (4.21), we have

$$\begin{aligned}
&Y_k\Big(\int_0^t \cdots \int_0^t (\varphi_{q,(u_1+\cdots+u_r)}^{\gamma} * \mathrm{d}\mu)\mathrm{d}u_1 \cdots \mathrm{d}u_r\Big) \\
&= \int_0^t \cdots \int_0^t \mathrm{e}^{-(q(k))^{\gamma}(u_1+\cdots+u_r)} Y_k(\mathrm{d}\mu)\mathrm{d}u_1 \cdots \mathrm{d}u_r \\
&= \Big(\int_0^t \cdots \int_0^t \mathrm{e}^{-(q(k))^{\gamma}(u_1+\cdots+u_r)} (-(q(k))^{\gamma})^r \mathrm{d}u_1 \cdots \mathrm{d}u_r\Big) Y_k f \\
&= (\mathrm{e}^{-(q(k))^{\gamma}} - 1)^r Y_k f \\
&= Y_k(\oplus^r T_{q,t}^{\gamma} f - f),
\end{aligned}$$

and hence,

$$\oplus^r T_{q,t}^{\gamma} f - f = \int_0^t \cdots \int_0^t (\varphi_{q,(u_1+\cdots+u_r)}^{\gamma} * \mathrm{d}\mu)\mathrm{d}u_1 \cdots \mathrm{d}u_r,$$

from which it follows that

$$\| \oplus^r T^{\gamma}_{q,t} f - f\|_1 \leqslant \|\mu\|_{\mathscr{M}} t^r = \mathscr{O}(t^r).$$

This completes the proof. □

5 Approximation by Booleans of generalized spherical Abel-Poisson and Weierstrass operators

We apply the results of Section 4 to two specific operators, the generalized spherical Abel-Poisson operator and the generalized spherical Weierstrass operator.

The generalized spherical Abel-Poisson operator on $\mathscr{X}$ (is also called the generalized Abel-Poisson sum or singular integral) is an operator on $\mathscr{X}$ such that (see [5])

$$V^{\gamma}_t f \sim \sum_{k=0}^{+\infty} \mathrm{e}^{-k^{\gamma} t} Y_k f = f * v^{\gamma}_t, \quad 0 < \gamma \leqslant 1,\ f \in \mathscr{X},$$

where $v^{\gamma}_t(\cos\theta)$ is the kernel given by

$$v^{\gamma}_t(\cos\theta) = \frac{1}{|\mathbb{S}^n|} \sum_{k=0}^{+\infty} \mathrm{e}^{-k^{\gamma} t} \frac{k+\lambda}{\lambda} C^{(\lambda)}_k(\cos\theta), \quad 0 \leqslant \theta \leqslant \pi.$$

For $\gamma = 1$, let $u = \mathrm{e}^{-t}$. Then V^{γ}_t reduces to the Abel-Poisson sum V^1_t of (1.3). For $r \in \mathbb{Z}_+$, the rth Boolean of V^{γ}_t on $\mathscr{X}$ is

$$\oplus^r V^{\gamma}_t f = f - (I - V^{\gamma}_t)^r f \sim \sum_{k=0}^{+\infty} (1 - (1 - \mathrm{e}^{-k^{\gamma} t})^r) Y_k f, \quad f \in \mathscr{X}.$$

The generalized spherical Weierstrass operator on $\mathscr{X}$ (is also called the generalized spherical Weierstrass singular integral) is given by (see [4])

$$W^{\kappa}_t f \sim \sum_{k=0}^{+\infty} \mathrm{e}^{-(k(k+2\lambda))^{\kappa} t} Y_k f = f * w^{\kappa}_t, \quad 0 < \kappa \leqslant 1,\ f \in \mathscr{X},$$

where w^{κ}_t is the kernel:

$$w^{\kappa}_t(\cos\theta) = \frac{1}{|\mathbb{S}^n|} \sum_{k=0}^{+\infty} \mathrm{e}^{-(k(k+2\lambda))^{\kappa} t} \frac{k+\lambda}{\lambda} C^{(\lambda)}_k(\cos\theta), \quad 0 \leqslant \theta \leqslant \pi.$$

For $r \in \mathbb{Z}_+$, the rth Boolean of W^{κ}_t on $\mathscr{X}$ is

$$\oplus^r W^{\kappa}_t f = f - (I - W^{\kappa}_t)^r f \sim \sum_{k=0}^{+\infty} (1 - (1 - \mathrm{e}^{-(k(k+2\lambda))^{\kappa} t})^r) Y_k f, \quad f \in \mathscr{X}.$$

The kernel of V_t^γ and W_t^κ are both positive, that is, for $0 \leqslant \theta \leqslant \pi$, $t > 0$, $0 < \gamma \leqslant 1$, and $0 < \kappa \leqslant 1$,

$$v_t^\gamma(\cos\theta) \geqslant 0, \quad w_t^\kappa(\cos\theta) \geqslant 0.$$

These were proved in [4,5,13]. Therefore, by Theorem 1, we have the following lemma.

Lemma 6 *The operators V_t^γ and W_t^κ form strongly continuous contraction semigroups of class $(\mathscr{C}_0)$ and are positive and regular exponential type multiplier operators with $q(x) = x$ and $q(x) = x(x+2\lambda)$, respectively, and admit the Bernstein type inequality, see* (4.9).

The approximation errors by these two operators are equivalent to the moduli of smoothness.

Theorem 3 *For $0 < \gamma \leqslant 1$, let $\{V_t^\gamma : 0 \leqslant t < +\infty\}$ be the generalized spherical Abel-Poisson operator. Then, for $r \in \mathbb{Z}_+$,*

$$\|\oplus^r V_t^\gamma f - f\|_{\mathscr{X}} \asymp \omega^{r\gamma}(f, t^{1/\gamma})_{\mathscr{X}}, \quad \forall\ f \in \mathscr{X},$$

where the constants depend only on n, r, γ, and $\mathscr{X}$.

Proof Denote by $\mathscr{V}^\gamma$ the infinitesimal generator of the semigroup $\{V_t^\gamma : 0 \leqslant t < +\infty\}$. By (4.8), for $(\mathscr{V}^\gamma)^r f \in \mathscr{X}$ and $r \in \mathbb{Z}_+$, we have

$$(\mathscr{V}^\gamma)^r f \sim \sum_{k=0}^{+\infty} (-k^\gamma)^r Y_k f.$$

By Lemma 6 and Theorem 2, we get

$$\| \oplus^r V_t^\gamma f - f\|_{\mathscr{X}} \asymp K_{(\mathscr{V}^\gamma)^r}(f, t^r)_{\mathscr{X}}.$$

In Lemma 2, let

$$a(x) = (-1)^{2/\gamma} x^2, \quad b(x) = -x(x+2\lambda), \quad \alpha = \frac{r\gamma}{2}.$$

Then

$$\lim_{x\to+\infty} \frac{(-1)^{2/\gamma} a(x)}{(-1)b(x)} = 1, \quad a(0) = b(0) = 0,$$

and

$$g(t) = 1 + 2\lambda t, \quad \frac{1}{g(t)} = (1+2\lambda t)^{-1}$$

are both in $C^{(2\lambda+2)}[0, +\infty)$. Then

$$K_{(\mathscr{V}^\gamma)^r}(f, t)_{\mathscr{X}} \asymp K_{(\Delta^*)^{r\gamma/2}}(f, t)_{\mathscr{X}}, \quad t > 0.$$

Thus, we have

$$\| \oplus^r V_t^\gamma f - f\|_{\mathscr{X}} \asymp K_{(\mathscr{V}\gamma)^r}(f,t^r)_{\mathscr{X}} \asymp K_{(\Delta^*)^{r\gamma/2}}(f,t^r)_{\mathscr{X}} \asymp \omega^{r\gamma}(f,t^{1/\gamma})_{\mathscr{X}},$$

where the equivalence uses (2.12) for which we take $\alpha = r\gamma$. This completes the proof. □

We have a similar equivalence for the generalized spherical Weierstrass operator as follows.

Theorem 4 *Let* $\{W_t^\kappa\colon 0 \leqslant t < +\infty\}$, $0 < \kappa \leqslant 1$, *be the generalized spherical Weierstrass operators on* $\mathscr{X}$. *Then, for any* $0 < \kappa \leqslant 1$ *and* $r \in \mathbb{Z}_+$,

$$\|\oplus^r W_t^\kappa f - f\|_{\mathscr{X}} \asymp \omega^{2r\kappa}(f,t^{1/(2\kappa)})_{\mathscr{X}}, \quad \forall\ f \in \mathscr{X},$$

where the constants depend only on n, r, κ, *and* $\mathscr{X}$.

In addition, the Booleans of V_t^γ and W_t^κ have the following saturation properties.

Theorem 5 *For reals* $0 < \gamma, \kappa \leqslant 1$ *and* $r \in \mathbb{Z}_+$, *we have*

(i) *the operators* $\oplus^r V_t^\gamma$ *are saturated with* $\mathscr{O}(t^r)$ *and their saturation class is* $\mathscr{H}(k^{r\gamma};\mathscr{X})$;

(ii) *the operators* $\oplus^r W_t^\kappa$ *are saturated with* $\mathscr{O}(t^r)$ *and their saturation class is* $\mathscr{H}((k(k+2\lambda))^{r\kappa};\mathscr{X})$;

(iii) *the operators* $\oplus^r V_t^\gamma$ *and* $\oplus^r W_t^\kappa$ *have the same saturation class if* $0 < \gamma = 2\kappa \leqslant 1$.

Proof (i) and (ii) follow from Lemma 6 and Theorem 2. (iii) follows from (i), (ii), and Remark 5 by setting

$$\psi_0(x) = (-1)^{1/(r\kappa)}x^2, \quad \varphi_0(x) = (-1)^{1/(r\kappa)}x(x+2\lambda), \quad s = r\kappa. \qquad □$$

6 Proofs

The proof of Lemma 1 uses the properties of the Gegenbauer-Stieltjes-coefficients.

Gegenbauer-Stieltjes-coefficients For $f \in \mathscr{L}^p(\mathbb{S}^n)$ and $\mu \in \mathscr{M}(\mathbb{S}^n)$, recall the convolution $f * \mathrm{d}\mu$, see (2.5). Let $\widehat{\mu}(k)$ be a sequence given by

$$\widehat{\mu}(k) := |\mathbb{S}^n| m(k,\lambda) \int_{\mathbb{S}^n} C_k^{(\lambda)}(\boldsymbol{x}\cdot\boldsymbol{y})\mathrm{d}\mu(\boldsymbol{y}) = |\mathbb{S}^n| m(k,\lambda) \int_0^\pi C_k^{(\lambda)}(\cos\theta)\mathrm{d}\mu^*(\theta),$$

where $m(k,\lambda)$ and μ^* are given by (2.8). Then we have

$$Y_k(f * \mathrm{d}\mu) = \frac{\lambda}{k+\lambda}\,\widehat{\mu}(k) Y_k(f).$$

We say that the sequence $\widehat{\mu}(k)$ is the Gegenbauer-Stietjes-coefficients of measure μ, see [3].

Remark 10 [3, Lemma 5.3.1] shows that a sequence is Gegenbauer-Stieltjes-coefficients of some zonal measure on $\mathbb{S}^n$ if and only if it belongs to $(\mathscr{M}, \mathscr{M})$.

Proof of Lemma 1 We first prove that for any real s,

$$\Big\{C_k^s = \frac{k+\lambda}{\lambda}\Big(\frac{\psi(k)}{\varphi(k)}\Big)^s,\ k=1,2,\ldots;\ C_0^s = \varphi(0)\Big\}$$

belongs to $(\mathscr{M}, \mathscr{M})$. Let k be any positive integer. That

$$g_1(t) = (g(t))^s \in \mathscr{C}^{2\lambda+2}[0,+\infty)$$

allows us to use Taylor's formula for $g_1(t)$ on $[0, 1/k]$ at $t = 0$, that is, there exists $0 < \xi_k < 1/k$ such that

$$\begin{aligned}
&\Big(\frac{\psi(k)}{\varphi(k)}\Big)^s \\
&= g_1\Big(\frac{1}{k}\Big) \\
&= g_1(0) + g_1^{(1)}(0)\frac{1}{k} + \cdots + \frac{g_1^{(2\lambda+1)}(0)}{(2\lambda+1)!}\Big(\frac{1}{k}\Big)^{2\lambda+1} + \frac{g_1^{(2\lambda+2)}(\xi_k)}{(2\lambda+2)!}\Big(\frac{1}{k}\Big)^{2\lambda+2}. \qquad (6.1)
\end{aligned}$$

By assumptions, $g_1^{(i)}(0)$, $i = 0, 1, \ldots, 2\lambda+1$, are constants depending only on φ, ψ, s, and n, and

$$|g_1^{(2\lambda+2)}(\xi_k)| \leqslant c_{\varphi,\psi,s,n}. \qquad (6.2)$$

Multiplying (6.1) by $(n+\lambda)/\lambda$, we may verify that the sequence of all the first terms

$$\frac{g_1(0)(k+\lambda)}{\lambda} = \frac{c_0(k+\lambda)}{\lambda}, \quad k = 1, 2, \ldots,$$

belongs to $(\mathscr{M}, \mathscr{M})$. From [1,2] for $\alpha > 0$, we know that $(k+\lambda)/k^\alpha$ are Gegenbauer-Stieltjes-coefficients of some measure in $\mathscr{M}$. For the last term of (6.1), by (6.2) and (2.9), we have

$$\begin{aligned}
\bigg|\frac{1}{|\mathbb{S}^n|}\sum_{k=1}^{+\infty}\frac{g_1^{(2\lambda+2)}(\xi_k)}{(2\lambda+2)!}\Big(\frac{1}{k}\Big)^{2\lambda+2}\frac{k+\lambda}{\lambda}C_k^{(\lambda)}(\cos\theta)\sin^{2\lambda}\theta\bigg| &\leqslant c_{\varphi,\psi,s,n}\sum_{k=0}^{+\infty}\frac{1}{k^2} \\
&< +\infty.
\end{aligned}$$

Thus, there exists $\mu_1 \in \mathscr{M}_\lambda(\mathbb{S}^n)$ such that the corresponding μ_1^* is

$$\mathrm{d}\mu_1^*(\theta) = \frac{1}{|\mathbb{S}^n|}\sum_{k=1}^{+\infty}\frac{g_1^{(2\lambda+2)}(\xi_k)}{(2\lambda+2)!}\Big(\frac{1}{k}\Big)^{2\lambda+2}\frac{k+\lambda}{\lambda}C_k^{(\lambda)}(\cos\theta)\sin^{2\lambda}\theta\mathrm{d}\theta.$$

It follows that the Gegenbauer-Stieltjes-coefficients of μ_1 are

$$\begin{aligned}\widehat{\mu}_1(j) &= |\mathbb{S}^n| m(j,\lambda) \int_0^{\pi} C_j^{(\lambda)}(\cos\theta)\mathrm{d}\mu_1^*(\theta)\\ &= \frac{j+\lambda}{\lambda}\,\frac{g_1^{(2\lambda+2)}(\xi_j)}{(2\lambda+2)!}\Big(\frac{1}{j}\Big)^{2\lambda+2}, \quad j=1,2,\ldots.\end{aligned}$$

By Remark 10, since

$$\Big\{\frac{k+\lambda}{\lambda}\,\frac{g_1^{(2\lambda+2)}(\xi_k)}{(2\lambda+2)!}\Big(\frac{1}{k}\Big)^{2\lambda+2}\Big\}_{k=1}^{+\infty}$$

are Gegenbauer-Stieltjes-coefficients of μ_1, we have

$$\Big\{C_k^s = \frac{k+\lambda}{\lambda}\Big(\frac{\psi(k)}{\varphi(k)}\Big)^s,\ k=1,2,\ldots;\ C_0^s = 0\Big\}$$

belongs to $(\mathscr{M},\mathscr{M})$. Then, by Remark 2, we get

$$\{C_k^s\}_{k=0}^{+\infty} \in (\mathscr{M},\mathscr{M}) = (\mathscr{C},\mathscr{C}) \subset (\mathscr{L}^p,\mathscr{L}^p), \quad 1 \leqslant p < +\infty,\ -\infty < s < +\infty. \tag{6.3}$$

Next, we prove

$$\mathscr{H}_1(\varphi^s;\mathscr{X}) = \mathscr{H}_1(\psi^s;\mathscr{X}).$$

We will only prove the case of $\mathscr{X} = \mathscr{L}^p(\mathbb{S}^n)$, $1 \leqslant p < +\infty$, and the proof for $\mathscr{X} = \mathscr{C}(\mathbb{S}^n)$ is analogous. For $f \in \mathscr{H}_1(\psi^s;\mathscr{L}^p(\mathbb{S}^n))$, $1 \leqslant p < +\infty$, $s \in \mathbb{R}$, there exists $g_1 \in \mathscr{L}^p(\mathbb{S}^n)$ such that

$$(\psi(k))^s Y_k f = Y_k g_1, \quad k = 0,1,2,\ldots. \tag{6.4}$$

Thus, we have

$$(\varphi(k))^s Y_k f = \Big(\frac{\varphi(k)}{\psi(k)}\Big)^s (\psi(k))^s Y_k f = \frac{\lambda}{k+\lambda}\, C_k^{-s} Y_k g_1, \quad k \geqslant 1.$$

It follows from (6.3) that $C_k^{-s} \in (\mathscr{L}^p,\mathscr{L}^p)$. Therefore, there exists $g_2 \in \mathscr{L}^p(\mathbb{S}^n)$ such that

$$\frac{\lambda}{k+\lambda}\, C_k^{-s} Y_k g_1 = Y_k g_2, \quad k = 0,1,2,\ldots, \tag{6.5}$$

that is,

$$(\varphi(k))^s Y_k f = Y_k g_2, \quad k = 1,2,\ldots.$$

In addition, by (6.4) and (6.5),

$$(\varphi(0))^s Y_0 f = 0 = Y_0 g_2,$$

Therefore,

$$f \in \mathscr{H}_1((\varphi(k))^s;\mathscr{X}).$$

Thus,

$$\mathscr{H}_1(\psi^s;\mathscr{X}) \subset \mathscr{H}_1(\varphi^s;\mathscr{X}).$$

Similarly, we may prove that

$$\mathscr{H}_1(\varphi^s;\mathscr{X}) \subset \mathscr{H}_1(\psi^s;\mathscr{X}).$$

Since $\varphi_0(k)$ differs $\varphi(k)$ only by a constant, we have

$$\mathscr{H}_1((\varphi_0)^s;\mathscr{X}) = \mathscr{H}_1(\varphi^s;\mathscr{X}) = \mathscr{H}_1(\psi^s;\mathscr{X}) = \mathscr{H}_1((\psi_0)^s;\mathscr{X}).$$

This completes the proof. □

Proof of Lemma 2 By Remark 4 and Lemma 1, we have

$$\mathscr{D}_1(\mathscr{A}^\alpha) = \mathscr{H}_1(a^\alpha;\mathscr{X}) = \mathscr{H}_1(b^\alpha;\mathscr{X}) = \mathscr{D}_1(\mathscr{B}^\alpha). \tag{6.6}$$

For $g \in \mathscr{D}_1(\mathscr{A}^\alpha)$, set

$$h := \sum_{k=0}^{+\infty} \Big(\frac{b(k)}{a(k)}\Big)^\alpha Y_k(\mathscr{A}^r g) = \sum_{k=0}^{+\infty} (b(k))^\alpha Y_k(g) \sim \mathscr{B}^r g.$$

We show that

$$\|h\|_{\mathscr{X}} \leqslant c_{a,b,\alpha,n_0} \|\mathscr{A}^r g\|_{\mathscr{X}}.$$

Setting

$$\psi(x) = \Big(\frac{b(x)}{a(x)}\Big)^\alpha, \quad x \in [0,+\infty),$$

we may verify that

$$|(\psi(x))^{(l+1)}| \leqslant c_{a,b,\alpha,l}(1+x)^{-(l+2)}, \quad x \geqslant 1,$$

from which it follows that

$$\begin{aligned} |\delta^{l+1}\psi(k)| &\leqslant \Big| \int_0^1 \cdots \int_0^1 \psi^{(l+1)}(x)\big|_{x=k+u_1+\cdots+u_{l+1}} \mathrm{d}u_1 \cdots \mathrm{d}u_{l+1} \Big| \\ &\leqslant c_{a,b,\alpha,l}(1+k)^{-(l+2)}. \end{aligned} \tag{6.7}$$

Thus, for $l > \lambda$, we have

$$\|h\|_{\mathscr{X}} \leqslant \sum_{k=0}^{+\infty} |\delta^{l+1}\psi(k)| \binom{k+l}{l} \|\tau_k^l(\mathscr{A}^\alpha g)\|_{\mathscr{X}} \leqslant c_{a,b,\alpha,l,\mathscr{X}} \|\mathscr{A}^\alpha g\|_{\mathscr{X}},$$

where the first inequality uses the summation by parts $(l+1)$ times, and the second follows from (2.10) and (6.7). Then,

$$\|\mathscr{B}^\alpha g\|_{\mathscr{X}} = \Big\| \sum_{k=0}^{+\infty} (b(k))^\alpha Y_k g \Big\|_{\mathscr{X}} \leqslant c_{a,b,\alpha,l,\mathscr{X}} \|\mathscr{A}^\alpha g\|_{\mathscr{X}}.$$

In the same way, we have

$$\|\mathscr{A}^\alpha g\|_{\mathscr{X}} \leqslant c_{a,b,\alpha,l,\mathscr{X}} \|\mathscr{B}^\alpha g\|_{\mathscr{X}}.$$

Therefore,

$$\|\mathscr{A}^\alpha g\|_{\mathscr{X}} \asymp \|\mathscr{B}^\alpha g\|_{\mathscr{X}}, \quad \forall\, g \in \mathscr{D}_1(\mathscr{A}^\alpha).$$

Hence, by (6.6), for $f \in \mathscr{X}$ and $t > 0$, we have

$$\begin{aligned} K_{\mathscr{A}^\alpha}(f,t) &= \inf_{g_1 \in \mathscr{D}_1(\mathscr{A}^\alpha)} \{\|f - g_1\|_{\mathscr{X}} + t^\alpha \|\mathscr{A}^\alpha g_1\|_{\mathscr{X}}\} \\ &\asymp \inf_{g_2 \in \mathscr{D}_1(\mathscr{B}^\alpha)} \{\|f - g_2\|_{\mathscr{X}} + t^\alpha \|\mathscr{B}^\alpha g_2\|_{\mathscr{X}}\} \\ &= K_{\mathscr{B}^\alpha}(f,t). \end{aligned}$$

This completes the proof. □

Proof of Lemma 3 First, we prove $\mathscr{D}_1(\mathscr{A}) \subset \mathscr{D}(\mathscr{A})$. Set $f \in \mathscr{D}_1(\mathscr{A})$ and $Mf \in \mathscr{X}$ such that

$$Mf = \sum_{k=0}^{+\infty} a_k Y_k f.$$

For $k \geqslant 0$ and each fixed $\boldsymbol{x} \in \mathbb{S}^n$, $Y_k(f;\boldsymbol{x})$ is a bounded linear functional on $\mathscr{X}$. It commutes with the Bochner integral, see (4.3). Then, for $k \geqslant 0$, we have

$$\begin{aligned} Y_k\left(\int_0^t T_u(Mf)\mathrm{d}u; \boldsymbol{x}\right) &= \int_0^t Y_k(T_u(Mf);\boldsymbol{x})\mathrm{d}u \\ &= \int_0^t \mathrm{e}^{a_k u} Y_k(Mf;\boldsymbol{x})\mathrm{d}u \\ &= \int_0^t \mathrm{e}^{a_k u} a_k Y_k(f;\boldsymbol{x})\mathrm{d}u \\ &= (\mathrm{e}^{a_k t} - 1) Y_k(f;\boldsymbol{x}) \\ &= Y_k(T_t f - f;\boldsymbol{x}). \end{aligned}$$

Hence, by the uniqueness theorem, we have

$$\frac{T_t - I}{t} f = \frac{1}{t}\int_0^t T_u(Mf)\mathrm{d}u, \quad \text{a.e.} \tag{6.8}$$

Since the semigroup $\{T(t)\colon 0 \leqslant t < +\infty\}$ is of class $(\mathscr{C}_0)$, by (1.8b) and (4.2), we have

$$\begin{aligned} \left\|\frac{T_t - I}{t} f - Mf\right\|_{\mathscr{X}} &= \left\|\frac{1}{t}\int_0^t (T_u(Mf) - Mf)\mathrm{d}u\right\|_{\mathscr{X}} \\ &\leqslant \sup_{0 \leqslant u < t} \|T_u(Mf) - Mf\|_{\mathscr{X}} \\ &\to 0, \quad t \to 0+. \end{aligned}$$

Therefore, $f \in \mathscr{D}(\mathscr{A})$, and

$$\mathscr{A} f = s\text{-}\lim_{t\to 0+} \frac{T_t - I}{t} f = Mf \sim \sum_{k=0}^{+\infty} a_k Y_k f.$$

Thus, $\mathscr{D}_1(\mathscr{A}) \subset \mathscr{D}(\mathscr{A})$.

On the other hand, for $r \in \mathbb{Z}_+$ and $f \in \mathscr{D}(\mathscr{A}^r)$, we have

$$\begin{aligned}\Big(\frac{\mathrm{e}^{a_k t} - 1}{t}\Big)^r Y_k(f;\boldsymbol{x}) &= Y_k\Big(\Big(\frac{T_t - I}{t}\Big)^r f;\boldsymbol{x}\Big)\\ &= Y_k\left(\int_0^t \cdots \int_0^t T_{u_1+\cdots+u_r}\mathscr{A}^r f \mathrm{d}u_1\cdots\mathrm{d}u_r;\boldsymbol{x}\right)\\ &= \int_0^t \cdots \int_0^t \mathrm{e}^{a_k(u_1+\cdots+u_r)}\mathrm{d}u_1\cdots\mathrm{d}u_r Y_k(\mathscr{A}^r f;\boldsymbol{x})\\ &= \Big(\frac{\mathrm{e}^{a_k t} - 1}{t}\Big)^r (a_k)^{-r} Y_k(\mathscr{A}^r f;\boldsymbol{x}), \quad k \geqslant 0,\end{aligned}$$

where the second equality follows from [6, Proposition 1.1.6]. Hence,

$$Y_k(\mathscr{A}^r f) = (a_k)^r Y_k f, \quad k \geqslant 0.$$

This means $f \in \mathscr{D}_1(\mathscr{A}^r)$. Thus, $\mathscr{D}(\mathscr{A}^r) \subset \mathscr{D}_1(\mathscr{A}^r)$.

To prove (4.5), we notice that

$$\begin{aligned}&Y_k\left(\int_0^t \cdots \int_0^t T_{u_1+\cdots+u_r} g \mathrm{d}u_1\cdots\mathrm{d}u_r;\boldsymbol{x}\right)\\ &\quad= \int_0^t \cdots \int_0^t \mathrm{e}^{a_k(u_1+\cdots+u_r)}(a_k)^r Y_k(f;\boldsymbol{x})\mathrm{d}u_1\cdots\mathrm{d}u_r\\ &\quad= (\mathrm{e}^{a_k t} - 1)^r Y_k(f;\boldsymbol{x})\\ &\quad= Y_k((T_t - I)^r f;\boldsymbol{x}) \qquad (6.9)\end{aligned}$$

which and the uniqueness theorem for the Fourier-Laplace series yield the result. This completes the proof. □

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